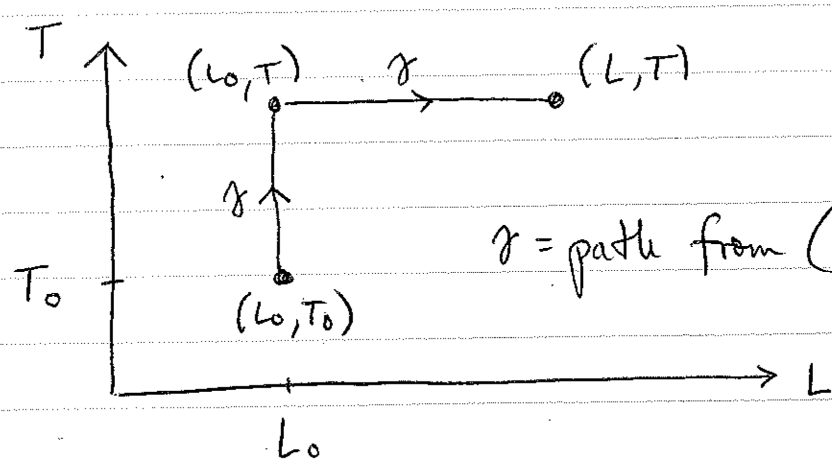
# Problem Set # 1, Solutions

1 a




$\gamma$ = path from $(L_0,T_0)$ to $(L,T)$

$$S(L,T) - S(L_0,T_0) = \int_\gamma \frac{đQ}{T}$$

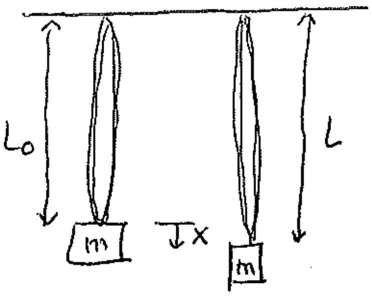
$$(L_0,T_0) \to (L_0,T) : \quad đQ = dU = c\, dT$$

$$(L_0,T) \to (L,T) : \quad dU = 0 \ \therefore \ đQ = -đW$$
$$= -\mathcal{F}\, dL$$
$$= -bT(L-L_0)\, dL$$

$$S(L,T) = S_0 + \int_{T_0}^{T} \frac{c\, dT}{T} - \int_{L_0}^{L} b(L-L_0)\, dL$$

$$= S_0 + c \ln\left(\frac{T}{T_0}\right) - \frac{b}{2}(L-L_0)^2$$

1 (b) $x = L - L_0$




$$m\ddot{L} = mg - bT(L - L_0)$$

$$\left.\begin{aligned} m\ddot{x} &= mg - bTx \\ p &= m\dot{x} \end{aligned}\right\} \rightarrow \boxed{\begin{aligned} \dot{x} &= p/m \\ \dot{p} &= mg - bTx \end{aligned}}$$

$$U = cT \rightarrow \dot{T} = \frac{1}{c}\dot{U} = \frac{1}{c}\left(\dot{W} + \dot{Q}\right)$$

$$= \frac{1}{c}\left[\cancel{F}\dot{L} - \alpha(T - T_0)\right]$$

$$\boxed{\dot{T} = \frac{1}{c}\left[bTx\dot{x} - \alpha(T - T_0)\right]}$$

(c) $\alpha = 0$ : $\dfrac{\dot{T}}{T} = \dfrac{b}{c}x\dot{x}$

$$\frac{d}{dt}\ln T = \frac{b}{2c}\frac{d}{dt}x^2$$

$$\ln\frac{T}{T_i} = \frac{b}{2c}\left(x^2 - x_i^2\right) \qquad \left(x_i, T_i = \text{initial values}\right)$$

$$T = T_i\, e^{\gamma(x^2 - x_i^2)/2} \qquad \gamma = \frac{b}{c}$$

$$\dot{p} = mg - bTx = mg - bT_i\, x\, e^{\gamma(x^2 - x_i^2)/2}$$

$$= mg - B\,x\,e^{\gamma x^2/2}$$

$$\left(B = bT_i\, e^{-\gamma x_i^2/2}\right)$$

$$\dot{p} = -\frac{\partial}{\partial x}\left(-mgx + \frac{B}{\gamma}\,e^{\gamma x^2/2}\right) = -\frac{\partial V}{\partial x}$$

$$\boxed{V(x) = -mgx + \frac{B}{\gamma}\,e^{\gamma x^2/2}}$$

(d) $\alpha \neq 0$ :

Conservation of total energy gives us :

$$\underbrace{\frac{d}{dt}\left(\frac{p^2}{2m} - mgx\right)}_{\dot{U}_{mass}} + \underbrace{\frac{d}{dt}(cT)}_{\dot{U}_{\text{rubber band}}} \underbrace{- \dot{Q}}_{\dot{U}_{\text{reservoir (air)}}} = 0$$

$$\therefore\quad \dot{Q} = \frac{d}{dt}\left(\frac{p^2}{2m} - mgx + cT\right)$$

2nd Law: $\underbrace{-\frac{\dot{Q}}{T}}_{\dot{S}_{air}} + \dot{S} \geq 0$

$$\dot{Q} - T_0 \dot{S} \leq 0$$

$$\frac{d}{dt}\Big(\underbrace{\frac{p^2}{2m} - mgx + cT - T_0 S}_{f(x,p,T)}\Big) \leq 0$$

Using expression for $S$ from part ⓐ:

$$f(x,p,T) = \frac{p^2}{2m} - mgx + cT \underbrace{- T_0 S_0}_{\text{irrelevant constant.}} - cT_0 \ln\frac{T}{T_0} + \frac{b}{2}T_0 x^2$$

Verify that $\frac{d}{dt} f(x,p,T) = 0$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial T}\dot{T}$$

$$= -mg\dot{x} + bT_0 x\dot{x} + \frac{p}{m}\dot{p} + c\dot{T} - c\frac{T_0}{T}\dot{T}$$

$$= \left(-mg + bT_0 x\right)\frac{p}{m} + \frac{p}{m}\left(mg - bTx\right)$$

$$+ \left[bTx\dot{x} - \alpha(T - T_0)\right]\left(1 - \frac{T_0}{T}\right)$$

$$= -\frac{\alpha}{T}(T - T_0)^2 \qquad \text{(after cancellations)}$$

$$\leq 0 \quad \checkmark$$

$$f(x,p,T) = \left(\frac{p^2}{2m}\right) + \left(\frac{b}{2} T_0 x^2 - mgx\right) + cT_0\left(\frac{T}{T_0} - \ln\frac{T}{T_0}\right) - T_0 S_0$$

The three terms in parentheses can be minimized independently.

The minimum value of $f$ occurs at:

$$p = 0$$
$$bT_0 x = mg \qquad \text{i.e.} \quad x = mg/bT_0$$
$$T = T_0$$

This describes a situation in which:

→ the mass is at rest
→ the tension in the rubber band is balanced by gravity
→ the temperature of the rubber band is the same as the surrounding air

. . . makes sense!
(Equilibrium!)

(e) (i) $0$ since $T_f = T_i = T_0$

(ii) $\Delta(KE) = 0$ since $p_i = p_f = 0$

$\Delta(PE) = -mg\,\Delta x = -m^2g^2/bT_0$

(iii) $+\dfrac{m^2g^2}{bT_0}$ to balance $\Delta(PE)$

(iv) $S_i = S_0$ since $(L_i, T_i) = (L_0, T_0)$

$$S_f = S_0 - \frac{b}{2}x_f^2 = S_0 - \frac{b}{2}\left(\frac{mg}{bT_0}\right)^2$$

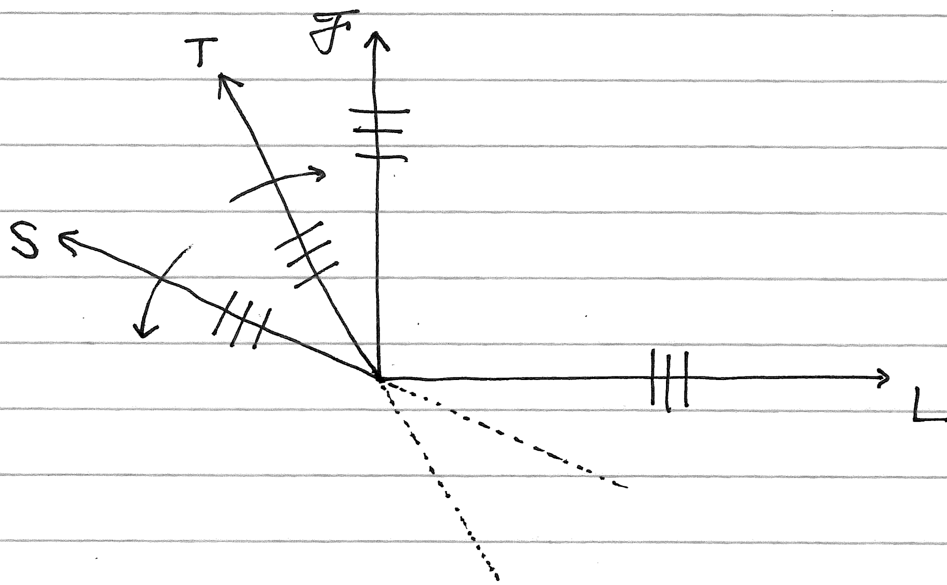
$$\Delta S = -\frac{1}{2T_0}\frac{m^2g^2}{bT_0}$$

(v) $\Delta S_{air} = \dfrac{Q_{air}}{T_0} = \dfrac{1}{T_0}\dfrac{m^2g^2}{bT_0}$

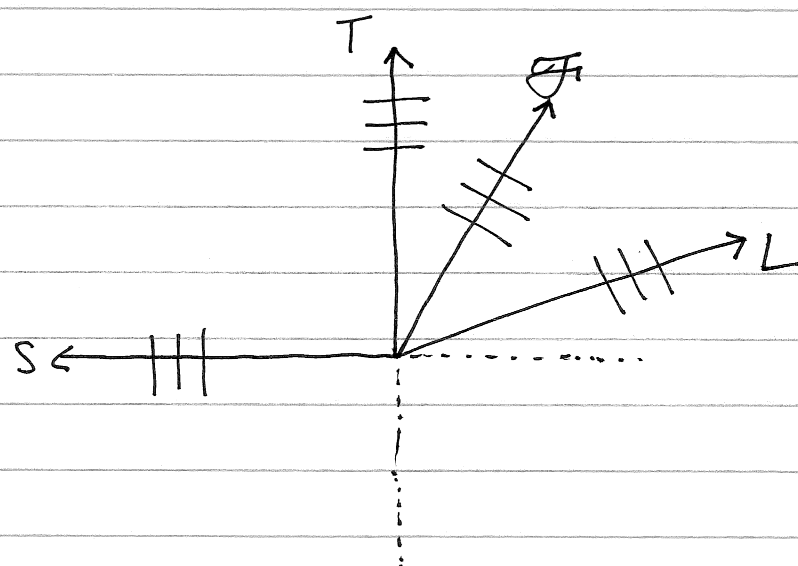
1st Law satisfied by construction (using iii)

2nd Law: $\Delta S + \Delta S_{air} = \dfrac{1}{2T_0}\dfrac{m^2g^2}{bT_0} > 0$ ✓

② ⓐ

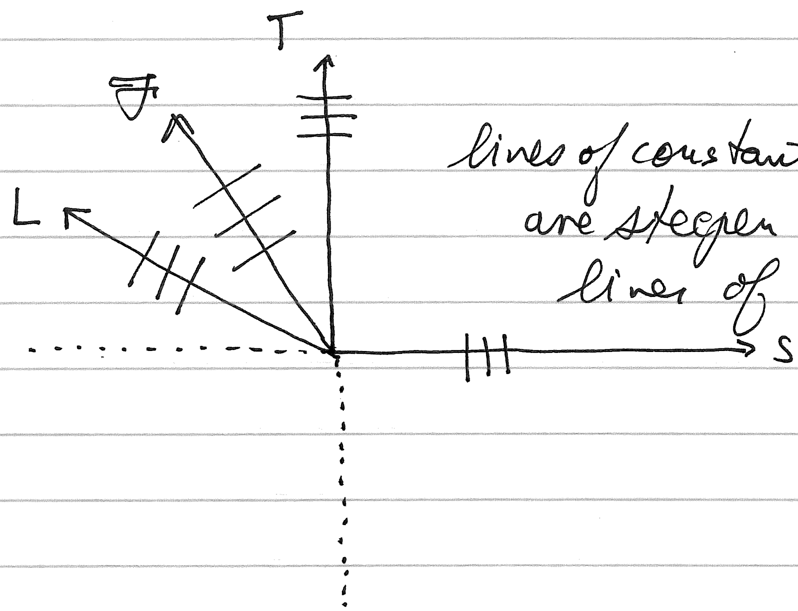



If we deform the plane by re-orienting the T & S axes along the directions indicated by the arrows, we get:



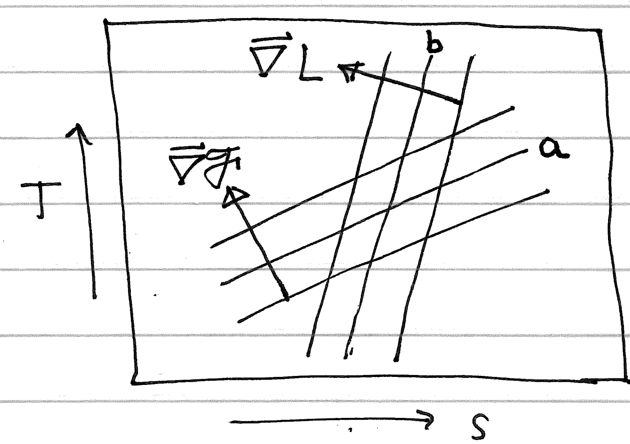

Now rotate 180° around the T-axis so that the S-axis points to the right:

→




lines of constant L
are steeper than
lines of constant $\mathcal{F}$

equivalently:




$$\left.\frac{\partial T}{\partial S}\right)_{\mathcal{F}} < \left.\frac{\partial T}{\partial S}\right)_{L}$$

slope of line "a"        slope of line "b"

therefore $\left.\frac{\partial S}{\partial T}\right)_{\mathcal{F}} > \left.\frac{\partial S}{\partial T}\right)_{L}$

$$T\left.\frac{\partial S}{\partial T}\right)_{\mathcal{F}} > T\left.\frac{\partial S}{\partial T}\right)_{L} \quad \therefore \boxed{C_{\mathcal{F}} > C_{L}}$$

(b) $$dU = T\,dS + \mathcal{F}\,dL$$

$$T = \left.\frac{\partial U}{\partial S}\right)_L \quad , \quad \mathcal{F} = \left.\frac{\partial U}{\partial L}\right)_S$$

$$\left.\frac{\partial T}{\partial L}\right)_S = \frac{\partial^2 U}{\partial L \partial S} = \frac{\partial^2 U}{\partial S \partial L} = \left.\frac{\partial \mathcal{F}}{\partial S}\right)_L$$

(c) $$T = T(L(\mathcal{F}, S), S)$$

$$\left.\frac{\partial T}{\partial S}\right)_{\mathcal{F}} = \left.\frac{\partial T}{\partial L}\right)_S \left.\frac{\partial L}{\partial S}\right)_{\mathcal{F}} + \left.\frac{\partial T}{\partial S}\right)_L$$

$$\therefore \left.\frac{\partial T}{\partial S}\right)_{\mathcal{F}} - \left.\frac{\partial T}{\partial S}\right)_L = \left.\frac{\partial T}{\partial L}\right)_S \left.\frac{\partial L}{\partial S}\right)_{\mathcal{F}}$$

Maxwell rel'n

$$= \left.\frac{\partial \mathcal{F}}{\partial S}\right)_L \left.\frac{\partial L}{\partial S}\right)_{\mathcal{F}}$$

cyclic identity $\left.\frac{\partial x}{\partial y}\right)_z \left.\frac{\partial y}{\partial z}\right)_x \left.\frac{\partial z}{\partial x}\right)_y = -1$

$$= -\left.\frac{\partial \mathcal{F}}{\partial S}\right)_L \left.\frac{\partial L}{\partial \mathcal{F}}\right)_S \left.\frac{\partial \mathcal{F}}{\partial S}\right)_L$$

$$= -\left[\left.\frac{\partial \mathcal{F}}{\partial S}\right)_L\right]^2 \cdot \left.\frac{\partial L}{\partial \mathcal{F}}\right)_S < 0$$

this must be positive by thermodynamic stability

thus $\left.\frac{\partial T}{\partial S}\right)_{\mathcal{F}} < \left.\frac{\partial T}{\partial S}\right)_L$ & we get

$C_{\mathcal{F}} > C_L$ just as in part (a)

(3) (a)
$$\Omega(E) = \int_{V^N} d^{3N}q \int d^{3N}p \ \theta\left(E - \frac{p^2}{2m}\right)$$

$$= V^{N} \cdot (2mE)^{3N/2} \int d^{3N}x \ \theta(1-x)$$

$$= \frac{(2\pi m E)^k}{k\Gamma(k)} V^N \qquad k = \frac{3N}{2}$$

using the expression for the volume of a 3N-dimensional sphere.

$$\Sigma = \frac{\partial \Omega}{\partial E} = \frac{(2\pi m)^k}{\Gamma(k)} V^N E^{k-1}$$

(b)
$$S = k_B \ln \frac{\Sigma}{N!}$$

$$S/k_B = N \ln V + \frac{3N}{2} \ln(2\pi m) + \left(\frac{3N}{2} - 1\right) \ln E$$

$$- \ln \Gamma\left(\frac{3N}{2}\right) - \ln N!$$

$$\cong N \ln V + \frac{3N}{2} \ln(2\pi m E) - \frac{3N}{2} \ln \frac{3N}{2} - N \ln N + \frac{3N}{2} + N$$

$$= N \ln\left(\frac{V}{N}\right) + \frac{3N}{2} \ln\left(\frac{4\pi m}{3} \frac{E}{N}\right) + \frac{5}{2} N$$

$$\frac{S}{k_B N} = \ln\left[\frac{V}{N} \cdot \left(\frac{4\pi m}{3} \frac{E}{N}\right)^{3/2}\right] + \frac{5}{2}$$

which is essentially the Sackur-Tetrode eqn, except the latter contains a factor $h^2$ that makes the argument of ln dimensionless.

(c) $\pi^c = \frac{1}{Z} e^{-\beta \mathbf{P}^2/2m}$ , $\mathbf{P}^2 = p_1^2 + \cdots p_N^2$

$$Z = V^N (2\pi m k_B T)^{3N/2}$$

$$-\ln \pi^c = \beta \frac{\mathbf{P}^2}{2m} + N \ln V + \frac{3N}{2} \ln(2\pi m k_B T)$$

$$\frac{S}{k_B} = -\int \pi^c \ln \pi^c - \ln N!$$

$$\cong \frac{3N}{2} + N \ln V + \frac{3N}{2} \ln(2\pi m k_B T) \cancel{-N} \ln N + N$$

$$\frac{S}{N k_B} = \ln\left[\frac{V}{N} \cdot (2\pi m k_B T)^{3/2}\right] + \frac{5}{2}$$

(d) $\rho^c(E) \propto e^{\ln \Sigma(E) - \beta E}$

$$\ln \Sigma(E) \cong \frac{3N}{2} \ln\left(\alpha \frac{E}{N}\right) \quad , \quad \alpha = V^{2/3} \cdot \frac{4\pi m e}{3}$$

$$\rho^c(E) \sim e^{-N \phi(\epsilon; T)}$$

$$\phi = \beta \epsilon - \frac{3}{2} \ln(\alpha \epsilon) \quad , \quad \epsilon = \frac{E}{N}$$

$$\frac{\partial^2 \phi}{\partial \epsilon^2} = \frac{3}{2\epsilon^2} > 0 \quad \checkmark$$